การวิเคราะห์การรบกวนทางควอนตัมสำหรับแอนฮาร์มอนิกออสซิลเลเตอร์กำลังสี่ในแบบเทียบเท่านิวตัน

Perturbative analysis of Newton-equivalent quantum quartic anharmonic oscillators

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บทคัดย่อ

ฮามิลโทเนียนแบบเทียบเท่านิวตันคือฮามิลโทเนียนที่พลวัตคลาสสิคตรงกับของฮามิลโทเนียนแบบมาตรฐาน ซึ่งในงานวิจัยนี้ได้หาค่าสเปกตรัมและฟังก์ชันคลื่นของระบบเชิงควอนตัมในแบบเทียบเท่านิวตันสำหรับศักย์แอนฮาร์มอนิกออสซิลเลเตอร์กำลังสี่ ในงานวิจัยได้แสดงผลตัวอย่างของการรบกวนในอันดับต่ำ ๆ ซึ่งในอันดับที่สูงขึ้นก็สามารถทำได้โดยการทำซ้ำ ผลการรบกวนในลิมิตที่ความเป็นแอนฮาร์มอนิกออสซิลเลเตอร์เป็นศูนย์นั้นสอดคล้องกับผลของฮาร์มอนิกออสซิลเลเตอร์แบบเทียบเท่านิวตันที่วิเคราะห์โดย Degasperis และ Ruijsenaars ในปี 2001 นอกจากนี้ในงานวิจัยยังได้ศึกษาฮามิลโทเนียนของแอนฮาร์มอนิกออสซิลเลเตอร์แบบเทียบเท่านิวตันซึ่งมีการจัดลำดับตัวดำเนินการในรูปแบบอื่น ๆ และแสดงผลการรบกวนซึ่งสอดคล้องกับความสัมพันธ์เชื่อมโยง ซึ่งอยู่ในกรอบความคิดที่กำหนดให้โดย Sasaki และ Odake ในปี 2009 และ 2011 ที่กล่าวถึงความสัมพันธ์ของระดับชั้นพลังงานสำหรับฮามิลโทเนียนที่มีการจัดลำดับตัวดำเนินการในแบบต่าง ๆ

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Abstract

Newton-equivalent Hamiltonians are Hamiltonians whose classical dynamics agree with those from the standard Hamiltonian. In this work, we perturbatively work out energy spectra and wavefunctions of quantum mechanical Newton-equivalent Hamiltonians with quartic-anharmonicity anharmonic oscillator potentials. We show example results for low order perturbations. Higher orders can also be obtained using further iterations. Our perturbative results in the limit of zero anharmonicity are consistent with the exact results of a Newton-equivalent simple harmonic oscillator analyzed by Degasperis and Ruijsenaars in 2001. We also study other orderings of the Newton-equivalent anharmonic oscillator Hamiltonians and show that our perturbative results are consistent with the intertwining relations, which are in the framework given by Sasaki and Odake in 2009 and in 2011, relating eigenenergies of different orderings of the Hamiltonians.

Keywords: Perturbation theory, anharmonic oscillator, energy spectra and wavefunctions, Newton-equivalent Hamiltonian, intertwining relations

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Introduction

In Hamiltonian analysis, the dynamics of a classical point particle of mass m moving in one dimension subject to an external potential $V(x)$ is usually described by starting from the Hamiltonian

$$H_E\,(x,p)=\frac{p^2}{2m}+V(x). \quad (1)$$

One type of alternative Hamiltonian giving the same dynamics was obtained by Degasperis and Ruijsenaars[1]. Such Hamiltonians would be called Newton-equivalent Hamiltonians. The idea of the construction was to look for the Hamiltonian as a product of a function of p and a function of x.This results in a one-parameter family of Hamiltonians:

$$H_c=4mc^2\cosh\left(\frac{p}{2mc}\right)\left(1+\frac{V(x)}{2mc^2}\right)^{1/2}, \quad (2)$$

where the parameter c was interpreted as the speed of light. Since c is a parameter, different c corresponds to different Hamiltonians, and hence describes different systems. In the limit $c\to\infty$, the Hamiltonian H_c reduce to H_E :

$$\lim_{c\to\infty}(H_c-4mc^2)=H_E. \quad (3)$$

Note that the interpretation that c is the speed of light is given only by Degasperis and Ruijsenaars[1]. However, throughout our work, we are not going to follow this interpretation. This is because c can take any value, and is not just limited to $3\times10^8\ m/s$. By “any values” we mean that c can be anything from $0\ m/s$ to $\infty\ m/s$. to For the Hamiltonians in eq. (2), different c corresponds to different Hamiltonians. For example, the Hamiltonian H_c with $c=3\times10^8\ m/s$ is differed from H_c with $c=2.4\times10^3\ m/s$, and both of them are differed from H_c with $c=9\times10^{10}\ m/s$, etc.

Another reason we are not going to view as the speed of light is that the theories that we discuss in this paper are non-relativistic. All the Hamiltonians in eq. (2) with any value of give rise, via Hamilton’s equation, to the usual Newton’s equation, which is exactly the same as that obtained from the Hamiltonian H_E in eq. (1). Note that Newton’s equation is non-relativistic, and that the parameter c does not appear anywhere. So the Hamiltonians in eq. (2) are not relativistic Hamiltonians.

Based on these reasons, we have made clear that we do not interpret c as a speed of light as in Degasperis and Ruijsenaars[1]. So it should not raise any concern when c takes values different from $3\times10^8\ m/s$. For us, we only view c as a parameter which happens to have the same symbol as speed of light and happens to have the unit m/s. Having commented on this point, let us now proceed.

The system for eq. (2) with a simple harmonic oscillator (SHO) potential as well as its quantization was investigated by Degasperis and Ruijsenaars[1]. Let us call this system a Newton-equivalent simple harmonic oscillator (NESHO). The complete energy spectra and wavefunctions were studied and determined. The result suggests that although classically, the whole family of Newton-equivalent Hamiltonitans describe the same dynamics, the quantum version does not need to agree. This observation is further confirmed by an investigation of Calogero and Degasperis[2].

There are several paths to generalize or give alternative viewpoints. Let us spell out some of them. The first one[3-4] discussed quantization of Newton-equivalent Hamiltonian which is also essentially of the form in eq. (2), but with an alternative operator ordering. In addition, Newton-equivalent Hamiltonian whose classical version takes the form[4], after an appropriate rescaling,

$$\widetilde{H}_c=4mc^2\times\sqrt{\left(1+\frac{V(x)}{2mc^2}\right)\frac{p^2}{4m^2c^2}+\frac{1}{1-\frac{V(x)}{2mc^2}}} \quad (4)$$

was investigated. In contrast to eq. (2), the dependencies on x and p are not separated. This Hamiltonian also connects to the standard one in the limit $c\to\infty$, in the same way as that of eq. (3). The quantization of the Hamiltonian in eq. (4) with simple harmonic potential was discussed. The energy spectra and wavefunctions were obtained.

For the second path[1], NESHO is in fact linked to Wigner quantum mechanics[5], which is considered a noncanonical quantization. A generalization of Degasperis and Ruijsenaars[1] in this framework is given in Blasiak, Horzela and Kapuscik[6].

In the third path[7-10], the idea is that quantum Hamiltonian obtained in Degasperis and Ruijsenaars[1] can be put in the form of discrete quantum mechanics. The analysis in this framework gives interesting consequences. As an example, this gives intertwining relations which allows one to obtain energy spectra of alternative orderings of some given Hamiltonians. Another example is that new orthogonal polynomials have to be used in order to write down wavefunctions.

In the fourth path[11-13], procedures to generate classes of Newton-equivalent Hamiltonians were developed and example Hamiltonians were generated. This path is developed quite recently, and so far only the classical aspects are discussed.

In this paper, we focus on a development which is mostly related to the third path. Although within the third path the procedure was already given in order to generate energy spectra and wavefunctions of some given classes of Hamiltonians, there is no guarantee that these can be given in closed forms. For example, even by using the standard Hamiltonian (the quantization of eq. (1)), quantum anharmonic oscillator needed to be solved using perturbation theory. So it is natural to expect that its Newton-equivalent version, called Newton-equivalent anharmonic oscillator (NEAHO), would also need assistances from perturbation theory. It is therefore our goal to study this. In fact, the only NEAHO that we are going to study in this paper are the ones with quartic anharmonicity. Nevertheless, we would still call this specific system as NEAHO.

Review of NESHO and its energy spectrum

Let us first review the construction of classical and quantum mechanical NESHO and their analyses[1].

By introducing $\beta = (2mc)^{-1}$ the Hamiltonian (2) becomes, after subtracted by a constant shift $1/(m\beta^2)$,

$$H_\beta(x,p) = \frac{1}{m\beta^2}\cosh(\beta p)\,(1 + 2m\beta^2 V(x))^{\frac{1}{2}} - \frac{1}{m\beta^2}. \tag{5}$$

The potential for a simple harmonic oscillator is given by

$$V(x) = \frac{1}{2}m\omega^2 x^2 \tag{6}$$

In this case, the Hamiltonian reads

$$H_\beta(x,p) = \frac{1}{m\beta^2}\cosh(\beta p) \times (1 + \beta^2 m^2\omega^2 x^2)^{\frac{1}{2}} - \frac{1}{m\beta^2}. \tag{7}$$

This is the Hamiltonian for the classical NESHO. One recovers the SHO Hamiltonian in the limit $\beta \to 0$.

By construction, the Newton's equation obtained from the classical NESHO Hamiltonian coincides with that from the standard SHO Hamiltonian. Therefore, the dynamics do not depend on the parameter β. In this sense, the whole family of the Hamiltonian (7) are equivalent.

Let us now turn to the quantum NESHO. The canonical quantization, by promoting $p \to \hat{p}, x \to \hat{x},$ of the Hamiltonian (7) is ambiguous as $\hat{p}$ and $\hat{x}$ no longer commute. The resulting Hamiltonians based on different ordering of the operators $\hat{p}$ and $\hat{x}$ could lead to a different theory. However, let us ignore an extensive discussion of this problem and simply focus on a particular choice of ordering. Consider a choice made by Degasperis and Ruijsenaars[1] which is based on physical insight of Ruijsenaars[14]. In this case, the NESHO Hamiltonian reads

$$\hat{H}_\beta \equiv \frac{1}{2\beta^2 m}\left(\hat{B}_+ \exp(\beta\hat{p})\,\hat{B}_- + (i \to -i)\right) - \frac{1}{\beta^2 m} + \frac{1}{2}\hbar\omega, \tag{8}$$

where $\hat{B}_+ = (1 + i\beta m\omega\hat{x})^{\frac{1}{2}},\ \hat{B}_- = (1 - i\beta m\omega\hat{x})^{\frac{1}{2}}.$

The terms in the second line of eq. (8) were not considered by Degasperis and Ruijsenaars[1]. However, we include them in order that the NESHO Hamiltonian reduces to the standard quantum mechanical SHO

Hamiltonian in the limit $\beta \to 0$. These terms simply provide a constant shift to the Hamiltonian, and hence it is safe to introduce them. The energy spectrum and wavefunctions of NESHO were also obtained by Degasperis and Ruijsenaars[1]. For this, the ladder operators were used. The idea is to start from the expression of lowering operator in terms of $\hat{x}$ and $\hat{p}$, and then replace $\hat{p}$ by $-im[\hat{x}, \hat{H}_\beta]/\hbar$. This gives

$$\hat{A}_\beta \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{1}{\hbar\omega}\left[\hat{x}, \hat{H}_\beta\right]\right) = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{2\beta m\omega}\left(\hat{B}_+ e^{\beta\hat{p}}\hat{B}_- - (i \to -i)\right)\right). \tag{9}$$

It satisfies the following commutation relations

$$[\hat{A}_\beta, \hat{H}_\beta] = \hbar\omega\hat{A}_\beta, \quad \left[\hat{A}_\beta^\dagger, \hat{H}_\beta\right] = -\hbar\omega\hat{A}_\beta^\dagger, \quad \left[\hat{A}_\beta, \hat{A}_\beta^\dagger\right] = \beta^2 m\left(\hat{H}_\beta - \frac{1}{\beta^2 m} + \frac{1}{2}\hbar\omega\right). \tag{10}$$

So $\hat{A}_\beta$ and $\hat{A}_\beta^\dagger$ are indeed ladder operators. The ground state wavefunction can be obtained from solving

$$\left\langle x\middle|\hat{A}_\beta\middle|\psi_0^{(\beta)}\right\rangle = 0, \tag{11}$$

whose solution is given by

$$\left\langle \mathrm{x}\middle|\psi_0^{(\beta)}\right\rangle = \psi_0^{(\beta)}(x) = \sqrt{\Gamma\left(\frac{1}{\beta^2 m\hbar\omega} + \frac{ix}{\hbar\beta}\right)\Gamma\left(\frac{1}{\beta^2 m\hbar\omega} - \frac{ix}{\hbar\beta}\right)}. \tag{12}$$

The energy $E_0^{(\beta)}$ of the ground state can be obtained from $\left\langle x\middle|\hat{H}_\beta\middle|\psi_0^{(\beta)}\right\rangle = E_0^{(\beta)}\psi_0^{(\beta)}(x)$. By using the second equality of eq. (10), it can be seen that energies for the excited states $|\psi_n^{(\beta)}\rangle = \left(\hat{A}_\beta^\dagger\right)^n|\psi_0^{(\beta)}\rangle$, are given by

$$E_n^{(\beta)} = \left(n + \frac{1}{2}\right)\hbar\omega. \tag{13}$$

The wavefunctions obtained are yet to be normalized. For this, it is convenient to make use of dimensionless quantities. More explicitly, we define

$$y = \sqrt{\frac{m\omega}{\hbar}}x, \qquad \tilde{\beta} = \beta\sqrt{m\hbar\omega}. \tag{14}$$

The outcome is that the normalized wavefunctions are given by

$$\tilde{\varphi}_n^{(\tilde{\beta})(DR)}(y) = p_n^{(\tilde{\beta})}(y)\sqrt{\frac{\Gamma\left(\tilde{\beta}^{-2} + i\tilde{\beta}^{-1}y\right)\Gamma\left(\tilde{\beta}^{-2} - i\tilde{\beta}^{-1}y\right)}{2^{1-2n-2/\tilde{\beta}^2}\, n!\,\pi\tilde{\beta}^{2n+1}\Gamma\left(n + 2/\tilde{\beta}^2\right)}}, \tag{15}$$

where $p_n^{(\tilde{\beta})}(y)$ satisfies the recurrence relation

$$p_{n+1}^{(\tilde{\beta})}(y) + \frac{\tilde{\beta}^2 n}{4}\left(n + \frac{2}{\tilde{\beta}^2} - 1\right)p_{n-1}^{(\tilde{\beta})}(y) = y p_n^{(\tilde{\beta})}(y) \tag{16}$$

for with $n = 2,3,4,\cdots$, with $p_0^{(\tilde{\beta})}(y) = 1,\ p_1^{(\tilde{\beta})}(y) = y$.

The wavefunctions (15) provide a good testing ground for the perturbative analyses to be carried out in this paper, where for NESHO we will expand the Hamiltonian in small β In the next section, we will write down the perturbation theory in the form which will be suitable for the study of multi-parameter Hamiltonians.

Extensions of time-independent non-degenerate perturbation theory

Standard Rayleigh-Schrödinger perturbation theory

In the standard Rayleigh-Schrödinger perturbation theory, one considers a Hamiltonian:

$$\hat{H} = \hat{H}^0 + \hat{H}^1, \tag{17}$$

which is a perturbed Hamiltonian of $\hat{H}^0$ with small $\hat{H}^1$ Suppose that all eigenstates and corresponding energy eigenvalues for $\hat{H}^0$ are known. One introduces a parameter λ and rewrite the perturbed Hamiltonian as

$$\hat{H} = \hat{H}^0 + \lambda\hat{H}^1. \tag{18}$$

The idea is to treat λ as a small parameter, perturbatively compute eigenstates and eigenenergies up to as many orders as required, and finally set $\lambda = 1$ at the end of the calculation.

Let $\{|\psi_n^0\rangle\}$ and $\{E_n^0\}$ be eigenstates and eigenenergies, respectively, for $\widehat{H}^0$ Let us focus on the case where energies are non-degenerate, and n is a discrete label of the states. Then, time-independent Schrödinger's equation is

$$\widehat{H}^0|\psi_n^0\rangle = E_n^0|\psi_n^0\rangle. \tag{19}$$

Similarly, let $\{|\psi_n\rangle\}$ and $\{E_n\}$ be eigenstates and eigenenergies, respectively, for $\widehat{H}$. Then

$$(\widehat{\mathrm{H}}^0 + \lambda\widehat{\mathrm{H}}^1)|\psi_{\mathrm{n}}\rangle = \mathrm{E_n}|\psi_{\mathrm{n}}\rangle. \tag{20}$$

Applying $\langle\psi_m^0|$ on this equation gives

$$E_m^0\langle\psi_m^0|\psi_n\rangle + \lambda\langle\psi_m^0|\,\widehat{H}^1|\psi_n\rangle = E_n\langle\psi_m^0|\psi_n\rangle. \tag{21}$$

There is a freedom to fix the normalization for $|\psi_n\rangle$. So let us define

$$|\varphi_n\rangle = \frac{|\psi_n\rangle}{\langle\psi_n^0|\psi_n\rangle}, \tag{22}$$

and consistently,

$$|\varphi_n^0\rangle = \frac{|\psi_n^0\rangle}{\langle\psi_n^0|\psi_n^0\rangle} = |\psi_n^0\rangle. \tag{23}$$

Therefore, $\langle\varphi_n^0|\varphi_n\rangle = 1$. With these definitions, eq. (21) becomes

$$E_m^0\langle\varphi_m^0|\varphi_n\rangle + \lambda\langle\varphi_m^0|\widehat{H}^1|\varphi_n\rangle = E_n\langle\varphi_m^0|\varphi_n\rangle. \tag{24}$$

It is convenient to separately consider the cases $m = n$ and $m \neq n$, giving

$$E_n = E_n^0 + \lambda\langle\varphi_n^0|\widehat{H}^1|\varphi_n\rangle, \tag{25}$$

and

$$E_m^0\langle\varphi_m^0|\varphi_n\rangle + \lambda\langle\varphi_m^0|\widehat{H}^1|\varphi_n\rangle = E_n\langle\varphi_m^0|\varphi_n\rangle, \tag{26}$$

where $m \neq n$. By expanding these two equations using

$$|\varphi_n\rangle = \sum_{\mu=0}^{\infty}\lambda^\mu|\varphi_n^\mu\rangle, \qquad E_n = \sum_{\mu=0}^{\infty}\lambda^\mu E_n^\mu, \tag{27}$$

and comparing coefficients of λ, one obtains, for $\mu = 1,2,3,\cdots$,

$$E_n^\mu = \langle\varphi_n^0|\widehat{H}^1|\varphi_n^{\mu-1}\rangle, \tag{28}$$

$$|\varphi_n^\mu\rangle = \sum_{m\neq n}\frac{|\varphi_m^0\rangle}{E_m^0 - E_n^0}$$

$$\times\left(\sum_{\nu=1}^{\mu-1}E_n^\nu\langle\varphi_m^0|\varphi_n^{\mu-\nu}\rangle - \langle\varphi_m^0|\widehat{H}^1|\varphi_n^{\mu-1}\rangle\right), \tag{29}$$

which should be solved recursively, starting from lower values of μ.

Perturbation theory for multi-parameter Hamiltonian

Instead of the Hamiltonian in the form of eq. (18), let us now turn to the case where the Hamiltonian has r parameters and is given as

$$\widehat{H} = \sum_{\mu_1=0}^{\infty}\sum_{\mu_2=0}^{\infty}\cdots\sum_{\mu_r=0}^{\infty}\lambda_1^{\mu_1}\lambda_2^{\mu_2}\cdots\lambda_r^{\mu_r}\widehat{H}^{\mu_1,\mu_2,\cdots,\mu_r}, \tag{30}$$

where $\widehat{H}^{0,0,\cdots,0}$ is the unperturbed Hamiltonian with known eigenenergies $E_n^{0,0,\cdots,0}$ and eigenstates $|\psi_n^{0,0,\cdots,0}\rangle$. Hamiltonians to be analyzed later have this form with $r = 1, 2$. So it is useful to give the analysis for perturbation theory on this kind of Hamiltonian with general r. Before proceeding, let us make use of multi-index notation $\vec{\mu} \equiv (\mu_1, \mu_2, \cdots, \mu_r)$, whose properties are

- $|\vec{\mu}| \equiv (\mu_1 + \mu_2 + \cdots + \mu_r)$,
- $\vec{\mu} \pm \vec{\nu} \equiv (\mu_1 \pm \nu_1, \mu_2 \pm \nu_2, \cdots, \mu_r \pm \nu_r)$,
- $\vec{\mu} > \vec{\nu} \Leftrightarrow \mu_i > \nu_i;\ 1 \leq i \leq r$,
- $\Lambda^{\vec{\mu}} \equiv \lambda_1^{\mu_1}\lambda_2^{\mu_2}\cdots\lambda_r^{\mu_r}$.

So the Hamiltonian can be rewritten as

$$\widehat{H} = \sum_{\vec{\mu}\geq\vec{0}}\Lambda^{\vec{\mu}}\widehat{H}^{\vec{\mu}}. \tag{31}$$

Let E_n and $|\psi_n\rangle$ be eigenenergies and eigenstates for $\widehat{H}$ We obtain

$$E_n = \langle \varphi_n^{\vec{0}} | \hat{H} | \varphi_n \rangle, \tag{32}$$

and

$$\left(E_m^{\vec{0}} - E_n\right)\left\langle \varphi_m^{\vec{0}} \middle| \varphi_n \right\rangle + \sum_{\vec{\mu}>\vec{0}} \left\langle \varphi_m^{\vec{0}} \middle| \Lambda^{\vec{\mu}} \hat{H}^{\vec{\mu}} \middle| \varphi_n \right\rangle = 0 \tag{33}$$

where $m \neq n$, and

$$|\varphi_n\rangle \equiv \frac{|\psi_n\rangle}{\left\langle \psi_n^{\vec{0}} \middle| \psi_n \right\rangle}, \qquad \left|\varphi_n^{\vec{0}}\right\rangle \equiv \left|\psi_n^{\vec{0}}\right\rangle. \tag{34}$$

By Expanding eigenenergies and scaled eigenstates as

$$E_n = \sum_{\vec{\mu}\geq\vec{0}} \Lambda^{\vec{\mu}} E_n^{\vec{\mu}}, \qquad |\varphi_n\rangle = \sum_{\vec{\mu}\geq\vec{0}} \Lambda^{\vec{\mu}} |\varphi_n^{\vec{\mu}}\rangle, \tag{35}$$

the equations (32)-(33) imply, for $\vec{\mu} \neq 0$,

$$\begin{aligned} E_n^{\vec{\mu}} &= \sum_{0<\vec{\nu}\leq\vec{\mu}} \langle \varphi_n^{\vec{0}} | \hat{H}^{\vec{\nu}} | \varphi_n^{\vec{\mu}-\vec{\nu}} \rangle, \\ |\varphi_n^{\vec{\mu}}\rangle &= \sum_{m\neq n} \sum_{0<\vec{\nu}\leq\vec{\mu}} \frac{\langle \varphi_m^{\vec{0}} | (\hat{H}^{\vec{\nu}} - E_n^{\vec{\nu}}) | \varphi_n^{\vec{\mu}-\vec{\nu}} \rangle}{E_n^{\vec{0}} - E_m^{\vec{0}}} |\varphi_m^{\vec{0}}\rangle. \end{aligned} \tag{36}$$

These two equations are solved recursively starting from smaller values of $|\vec{\mu}|$, and stops when the desired precision is achieved.

The main analyses in this paper make use of one-parameter and two-parameter Hamiltonians. So the cases $r = 1,2$ will be useful for later presentations in this paper. Let us thus write down the results for these cases for convenience.

One-parameter Hamiltonian

Let us consider a Hamiltonian with one parameter λ:

$$\hat{H} = \sum_{\mu=0}^{\infty} \lambda^{\mu} \hat{H}^{\mu}, \tag{37}$$

where $\hat{H}^0$ is the Hamiltonian whose normalized eigenstates $|\varphi_n^0\rangle$ and eigenenergies E_n^0 are known. The energies and eigenstates of $\hat{H}$ are given by

$$E_n = \sum_{\mu=0}^{\infty} \lambda^{\mu} E_n^{\mu}, \qquad |\varphi_n\rangle = \sum_{\mu=0}^{\infty} \lambda^{\mu} |\varphi_n^{\mu}\rangle, \tag{38}$$

whose coefficients for each order of λ can be obtained from the recurrence relations

$$\begin{aligned} E_n^{\mu} &= \sum_{\nu=1}^{\mu} \langle \varphi_n^{0} | \hat{H}^{\nu} | \varphi_n^{\mu-\nu} \rangle, \\ |\varphi_n^{\mu}\rangle &= \sum_{\nu=1}^{\mu} \sum_{m\neq n} \frac{\langle \varphi_m^{0} | \hat{H}^{\nu} - E_n^{\nu} | \varphi_n^{\mu-\nu} \rangle}{E_n^{0} - E_m^{0}} |\varphi_m^{0}\rangle, \end{aligned} \tag{39}$$

where $\mu = 1, 2, 3, \cdots$.

Two-parameter Hamiltonian

Let us now turn to the Hamiltonian

$$\hat{H} = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \lambda_1^{\mu} \lambda_2^{\nu} \hat{H}^{\mu,\nu}, \tag{40}$$

where λ_1, and λ_2 are parameters. In this case, the unperturbed Hamiltonian is written as $\hat{H}^{0,0}$ with known eigenenergies $E_n^{0,0}$, and normalized eigenstates $|\varphi_n^{0,0}\rangle$. The eigenenergies and eigenstates of H are given by

$$\begin{aligned} E_n &= \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \lambda_1^{\mu} \lambda_2^{\nu} E_n^{\mu,\nu}, \\ |\varphi_n\rangle &= \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \lambda_1^{\mu} \lambda_2^{\nu} |\varphi_n^{\mu,\nu}\rangle, \end{aligned} \tag{41}$$

where, for $(\mu, \nu) \neq (0,0)$,

$$E_n^{\mu,\nu} = \sum_{\substack{\rho=0 \\ (\rho,\eta)\neq(0,0)}}^{\mu} \sum_{\eta=0}^{\nu} \langle \varphi_n^{0,0} | \hat{H}^{\rho,\eta} | \varphi_n^{\mu-\rho,\nu-\eta} \rangle, \tag{42}$$

$$|\varphi_n^{\mu,\nu}\rangle = \sum_{\substack{\rho=0 \\ (\rho,\eta)\neq(0,0)}}^{\mu} \sum_{\eta=0}^{\nu} \sum_{m\neq n} \frac{\langle \varphi_m^{0,0} | \hat{H}^{\rho,\eta} - E_n^{\rho,\eta} | \varphi_n^{\mu-\rho,\nu-\eta} \rangle}{E_n^{0,0} - E_m^{0,0}} |\varphi_m^{0,0}\rangle. \tag{43}$$

Energy spectrum of NEAHO from perturbation theory

As a direct extension to NESHO, let us turn to NEAHO. That is, we consider a Hamiltonian whose Hamilton's equations of its classical version agree with Newton's equation for a system with Hamiltonian

$$H_c^0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \frac{1}{4}\alpha x^4. \quad (44)$$

From eq. (2), we see that the classical version of the required Hamiltonian is given by

$$H_c = \frac{\cosh \beta p}{\beta^2 m} \times \sqrt{1 + 2m\left(\frac{1}{2}m\omega^2 x^2 + \frac{1}{4}\alpha x^4\right)\beta^2} - \frac{1}{\beta^2 m}. \quad (45)$$

Let us canonically quantize this Hamiltonian by choosing the operator ordering in the spirit of Degasperis and Ruijsenaars[1]. This gives

$$\langle x|\hat{H}_A|\phi\rangle = \left(\left(C_+ \frac{e^{-i\hbar\beta\partial_x}}{2\beta^2 m} C_- + (i \to -i)\right) - \frac{1}{\beta^2 m} + \frac{1}{2}\hbar\omega\right)\phi(x), \quad (46)$$

where $C_\pm = \left(1 \pm i\beta\sqrt{2mV(x)}\right)^{\frac{1}{2}}$, ∂_x acts on everything on its right, and

$$V(x) = \frac{1}{2}m\omega^2 x^2 + \frac{1}{4}\alpha x^4. \quad (47)$$

Before presenting the analysis of NEAHO, it is a good idea to revisit NESHO, but using perturbation theory this time. This is in order to make sure that we have the technique under control.

The analysis of NESHO using perturbation theory

The Hamiltonian for NESHO, acting on a wavefunction $\phi(x)$, is given by

$$\langle x|\hat{H}|\phi\rangle = \left(\left(\hat{B}_+ \frac{e^{-i\hbar\beta\partial_x}}{2\beta^2 m}\hat{B}_- + (i \to -i)\right) - \frac{1}{\beta^2 m} + \frac{1}{2}\hbar\omega\right)\phi(x). \quad (48)$$

Let us define the $y-$ space representation accordingly by using eq. (14) and denote wavefunctions in $y-$ space representation as hatted notations, whereas those in $x-$ space as unhatted ones. For example,

$$\phi(x) \equiv \langle x|\phi\rangle, \quad (49)$$

whereas

$$\hat{\phi}(y) \equiv \langle y|\phi\rangle. \quad (50)$$

The hatted wavefunctions should not be confused with the operators. By using dimensionless quantities, the equation (48) reads

$$\langle y|\hat{H}|\phi\rangle = \left(\left(\tilde{B}_+ \frac{e^{-i\tilde{\beta}\partial_y}}{2\tilde{\beta}^2}\tilde{B}_- + (i \to -i)\right) - \frac{1}{\tilde{\beta}^2} + \frac{1}{2}\right)\hbar\omega\hat{\phi}(y), \quad (51)$$

where $\tilde{B}_+ = (1 + i\tilde{\beta}y)^{\frac{1}{2}}$, $\tilde{B}_- = (1 - i\tilde{\beta}y)^{\frac{1}{2}}$. Expanding gives $\hat{H}$ as given in eq. (37), where $\lambda = \tilde{\beta}^2$. Eigenstates for standard SHO Hamiltonian are given by

$$\hat{\varphi}_n^0(y) = \sqrt{\frac{m\omega}{\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}T_n(y)e^{-y^2/2}, \quad (52)$$

where

$$T_n(y) = (-1)^n e^{y^2}\frac{d^n}{dy^n}e^{-y^2} \quad (53)$$

are the (physicists') Hermite polynomials. The eigenstates $\hat{\varphi}_n^0(y)$ satisfy

$$\int dy \left(\frac{\hbar}{m\omega}\right)^{\frac{3}{2}}\left(\hat{\varphi}_l^0(y)\right)^* \hat{\varphi}_n^0(y) = \delta_{ln}. \quad (54)$$

These setups allow us to compute the eigenenergies and eigenstates. To demonstrate the calculation steps, let us show intermediate results up to order λ^2. First consider

$$\begin{aligned}\frac{\langle y|\hat{H}^1|\varphi_n^0\rangle}{\hbar\omega} = &-\frac{1}{12}\sqrt{(n)_4}\hat{\varphi}_{n-4}^0(y) \\ &-\frac{1}{6}(n-2)\sqrt{(n)_2}\,\hat{\varphi}_{n-2}^0(y) \\ &-\frac{1}{6}n\sqrt{(n+2)_2}\,\hat{\varphi}_{n+2}^0(y) \\ &-\frac{1}{12}\sqrt{(n+4)_4}\,\hat{\varphi}_{n+4}^0(y),\end{aligned} \quad (55)$$

where $(x)_k = \frac{x!}{(x-k)!}$ is the Pochhammer symbol.

So

$$\frac{\langle\varphi_m^0|\hat{H}^1|\varphi_n^0\rangle}{\hbar\omega} = -\frac{1}{12}\sqrt{(n)_4}\,\delta_{m,n-4} - \frac{1}{6}(n-2)\sqrt{(n)_2}\delta_{m,n-2} - \frac{1}{6}n\sqrt{(n+2)_2}\,\delta_{m,n+2} - \frac{1}{12}\sqrt{(n+4)_4}\,\delta_{m,n+4}. \quad (56)$$

So by using (39), we obtain

$$E_n^1 = 0. \quad (57)$$

Next, by using eq.(56) for $m \neq n$, we obtain

$$\hat{\varphi}_n^1(y) = -\frac{1}{48}\sqrt{(n)_4}\hat{\varphi}_{n-4}^0(y) - \frac{1}{12}(n-2)\sqrt{(n)_2}\,\hat{\varphi}_{n-2}^0(y) + \frac{1}{12}n\sqrt{(n+2)_2}\,\hat{\varphi}_{n+2}^0(y) + \frac{1}{48}\sqrt{(n+4)_4}\hat{\varphi}_{n+4}^0(y). \quad (58)$$

To rewrite this more compactly, let us define

$$\breve{\varphi}_{n+k}^0(y) \equiv \hat{\varphi}_{n+k}^0(y)\frac{\sqrt{\left(n+\frac{k+|k|}{2}\right)|k|}}{k^2}, \quad (59)$$

and the bracket

$$\left[f(n)\breve{\varphi}_{n+k}^0(y)\right]_\pm \equiv f(n)\breve{\varphi}_{n+k}^0(y) \pm f(n-k)\breve{\varphi}_{n-k}^0(y). \quad (60)$$

So eq. (58) is rewritten as

$$\hat{\varphi}_n^1(y) = \frac{1}{3}[\breve{\varphi}_{n+4}^0(y) + n\breve{\varphi}_{n+2}^0(y)]_-. \quad (61)$$

Let us now turn to the second order. Using eq. (56), we obtain

$$\langle\varphi_n^0|\hat{H}^1|\varphi_n^1\rangle = -\frac{\hbar\omega}{72}(10n^3 - 3n^2 + 11n + 3), \quad (62)$$

and

$$\langle\varphi_n^0|\hat{H}^2|\varphi_n^0\rangle = \frac{\hbar\omega}{72}(10n^3 - 3n^2 + 11n + 3). \quad (63)$$

Inserting eq. (62) and eq. (63) into eq. (39) gives

$$E_n^2 = \langle\varphi_n^0|\hat{H}^1|\varphi_n^1\rangle + \langle\varphi_n^0|\hat{H}^2|\varphi_n^0\rangle = 0. \quad (64)$$

Next, using eq. (39) we obtain

$$\hat{\varphi}_n^2(y) = \left[\frac{1}{72}\breve{\varphi}_{n+8}^0(y) + \frac{n+2}{16}\breve{\varphi}_{n+6}^0(y) + \frac{n(n+2)}{18}\breve{\varphi}_{n+4}^0(y) - \frac{(n+1)(n^2+2n+12)}{144}\breve{\varphi}_{n+2}^0(y)\right]_+ - \left[\frac{3}{20}\breve{\varphi}_{n+6}^0(y) + \frac{1}{60}(14n+15)\breve{\varphi}_{n+4}^0(y) + \frac{1}{24}(2n+1)n\breve{\varphi}_{n+2}^0(y)\right]_-. \quad (65)$$

We have followed the steps outlined above up to order λ^5. The results are

$$E_n = \hbar\omega\left(n+\frac{1}{2}\right) + \mathcal{O}(\lambda^6), \quad (66)$$

$$\hat{\varphi}_n(y) = \sum_{\mu=0}^{5}\lambda^\mu\hat{\varphi}_n^\mu(y) + \mathcal{O}(\lambda^6), \quad (67)$$

where $\hat{\varphi}_n^0(y)$, $\hat{\varphi}_n^1(y)$, and $\hat{\varphi}_n^2(y)$ are respectively as given in eq. (52), (58), and (65) while $\hat{\varphi}_n^3(y)$, $\hat{\varphi}_n^4(y)$, and $\hat{\varphi}_n^5(y)$ are given by long expressions which we choose not to write down as they do not give much insights. In fact, it is more important to verify or justify that the perturbative method we used is working as expected. For this, we will present the check our perturbative expansion against the exact result[1].

As discussed earlier, the energy spectrum for NESHO Hamiltonian[1] is, after a suitable shift using a constant, as given by eq. (13). So it can immediately be seen that our perturbative result in eq. (66) agrees with that of Degasperis and Ruijsenaars[1] for any n and up to order λ^5, or in terms of β, order β^{10}.

Next, let us check the eigenstates. In fact, eq. (67) is not yet suitable to compare with Degasperis and Ruijsenaars[1] as it needs to be normalized in the same way. More explicitly, we compute

$$\tilde{\varphi}_n(y) = \frac{\hat{\varphi}_n(y)}{\sqrt{\int_{-\infty}^{\infty} dy|\hat{\varphi}_n(y)|^2}} + \mathcal{O}(\lambda^6), \quad (68)$$

which is Taylor expanded around $\lambda = 0$ up to order λ^5. We compare this with the exact result of

Degasperis and Ruijsenaars[1]. This is as given by eq. (18). Let us relabel the wavefunctions $\tilde{\varphi}_n^{(\tilde{\beta})(DR)}(y)$ as $\tilde{\varphi}_n^{(DR)}(y)$ in order to make the notations less cluttered. Then make an expansion around $\tilde{\beta} = 0$ up to order at least $\tilde{\beta}^{10}$, and then write in terms of $\lambda = \tilde{\beta}^2$. The expansion can be carried out by making use of Stirling's series, which is the asymptotic expansion for large of

$$\Gamma(N+1) \sim \sqrt{2\pi N}\left(\frac{N}{e}\right)^N \sum_{k=0}^{\infty} \frac{a_k}{N^k}, \tag{69}$$

with[15]

$$a_k = \frac{(2k)!}{2^k k!} \sum_{i=0}^{2k} \binom{k+i-\frac{1}{2}}{i} \binom{3k+\frac{1}{2}}{2k-i} 2^i \times \sum_{j=0}^{i} \binom{i}{j} \frac{(-1)^j}{(2k+i+j)!} \times \sum_{l=0}^{j} (-1)^l \binom{j}{l} (j-l)^{2k+i+j}, \tag{70}$$

and $a_0 = 1$. For example, by using the above prescriptions, the expansion of the ground state wavefunction reads

$$\tilde{\varphi}_0^{(DR)}(y) = \frac{e^{-\frac{y^2}{2}}}{\pi^{\frac{1}{4}}} \times \left(1 + \frac{3 - 12y^2 + 4y^4}{48}\lambda + \cdots\right), \tag{71}$$

where ... are higher orders in λ which can be computed to any desired order. This result is checked against ours in eq. (68) for $n = 0$, and is shown to agree to order λ^5. In fact, we have explicitly verified the expression

$$\tilde{\varphi}_n(y) = \tilde{\varphi}_n^{(DR)}(y) + \mathcal{O}(\lambda^6) \tag{72}$$

for $n = 0, 1, 2, \cdots, 8$. Higher values of n can also be checked, and we expect that the expression remains true for these cases. Although we do not have an explicit proof that this is indeed valid for any given n the result should be sufficient to convince that perturbative calculation is working as expected. Let us then proceed to perturbatively analyze Newton-equivalent Hamiltonians for anharmonic oscillator.

The analysis of NEAHO using perturbation theory

NEAHO Hamiltonian, acting on a wavefunction $\phi(x)$, is given by

$$\langle x|\hat{H}_A|\phi\rangle = \left(\left(C_+ \frac{e^{-i\hbar\beta\partial_x}}{2\beta^2 m} C_- + (i \to -i)\right) - \frac{1}{\beta^2 m} + \frac{1}{2}\hbar\omega\right)\phi(x), \tag{73}$$

Note that in the limit of $\beta \to 0$, the equation (73) becomes

$$\langle x|\hat{H}_A|\,\phi\rangle \to \left(\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2x^2 + \frac{1}{4}\alpha x^4 + \frac{\omega\hbar}{2} - \frac{\hbar(m\omega^2+\alpha x^2)}{\sqrt{2m(2m\omega^2+\alpha x^2)}}\right)\phi(x), \tag{74}$$

which is differed from the standard Hamiltonian of anharmonic oscillator with potential in eq. (47). In fact, the extra terms are artefacts from the choice of operator ordering, which is considered a quantum effect. If one sets $\hbar \to 0$, thus turning off the quantum effect, then the Hamiltonian (81) is reduced to the expected classical Hamiltonian for anharmonic oscillator.

By using dimensionless quantities, the equation (74) becomes

$$\langle x|\hat{H}_A|\,\phi\rangle = \left(\left(D_+ \frac{e^{-i\tilde{\beta}\partial_y}}{2\tilde{\beta}^2} D_- + (i \to -i)\right) - \frac{1}{\tilde{\beta}^2} + \frac{1}{2}\right)\hbar\omega\hat{\phi}(y), \tag{75}$$

where $D_\pm = \left(1 \pm i\tilde{\beta}\sqrt{y^2+\tilde{\alpha}y^4}\right)^{\frac{1}{2}}$, $\tilde{\alpha} = \frac{\hbar}{2m^2\omega^3}\alpha$.

By expanding in $\tilde{\alpha}$ and $\tilde{\beta}$, the equation (75) takes the form

$$\langle y|\hat{H}_A|\phi\rangle = \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty} \lambda_1^{\mu}\lambda_2^{\nu}\langle y|\hat{H}_A^{\mu,\nu}|\phi\rangle, \tag{76}$$

where $\lambda_1 = \tilde{\alpha}$, $\lambda_2 = \tilde{\beta}^2$, and

$$\frac{\langle y|\hat{H}_A^{0,0}|\phi\rangle}{\hbar\omega} = \left(\frac{y^2}{2} - \frac{\partial_y^2}{2}\right)\hat{\phi}(y), \tag{77}$$

$$\frac{\langle y|\hat{H}_A^{0,1}|\phi\rangle}{\hbar\omega} = \frac{1}{24}\big(-6(y^2-1)\partial_y^2 + \partial_y^4 - 12y\partial_y - 3(y^2-1)^2\big)\hat{\phi}(y), \tag{78}$$

$$\frac{\langle y|\hat{H}_A^{1,0}|\phi\rangle}{\hbar\omega} = \left(\frac{y^4}{2} - \frac{3y^2}{4}\right)\hat{\phi}(y), \tag{79}$$

$$\frac{\langle y|\hat{H}_A^{1,1}|\phi\rangle}{\hbar\omega} = \frac{1}{8}\big(y^4(5-2\partial_y^2) - 8y^3\partial_y + 3y^2(\partial_y^2-3) + 6y\partial_y - 2y^6 + 2\big)\hat{\phi}(y), \tag{80}$$

etc.

By using eq. (41)-(43), with $E_n^{(0,0)}$ and $|\varphi_n^{0,0}\rangle$ being eigenenergies and eigenstates for the standard simple harmonic oscillator Hamiltonian, we obtain the eigenenergies for NEAHO Hamiltonian (75) as

$$\frac{E_{An}}{\hbar\omega} = n + \frac{1}{2} + E_{An}^{(1)}\tilde{\alpha} + E_{An}^{(2)}\tilde{\alpha}^2 + \mathcal{O}(\tilde{\alpha}^3), \tag{81}$$

where

$$E_{An}^{(1)} = \frac{3n^2}{4} + \frac{1}{4}n^3\,\tilde{\beta}^2 + \mathcal{O}(\tilde{\beta}^8),$$
$$E_{An}^{(2)} = -\frac{17}{32}(n+2n^3) - \frac{3n^2}{64}(17+25n^2)\tilde{\beta}^2 - \frac{n^3}{64}(17+9n^2)\tilde{\beta}^4 + \mathcal{O}(\tilde{\beta}^8),$$

and obtain the wavefunctions for NEAHO Hamiltonian (75) as

$$\tilde{\varphi}_n(y) = \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}\tilde{\varphi}_n^{\mu,\nu}(y)\tilde{\alpha}^{\mu}\tilde{\beta}^{2\nu}, \tag{82}$$

with

$$\left(\frac{m\omega}{\hbar}\right)^{\frac{3}{4}}\tilde{\varphi}_n^{0,0}(y) = \hat{\varphi}_n^0(y), \tag{83}$$

$$\left(\frac{m\omega}{\hbar}\right)^{\frac{3}{4}}\tilde{\varphi}_n^{0,1}(y) = \frac{1}{3}[\check{\varphi}_{n+4}^0(y) + n\check{\varphi}_{n+2}^0(y)]_-, \tag{84}$$

$$\left(\frac{m\omega}{\hbar}\right)^{\frac{3}{4}}\tilde{\varphi}_n^{1,0}(y) = -\left[\frac{1}{2}\check{\varphi}_{n+4}^0(y) + \frac{1}{4}(4n+3)\check{\varphi}_{n+2}^0(y)\right]_-, \tag{85}$$

etc.

$$\left(\frac{m\omega}{\hbar}\right)^{\frac{3}{4}}\tilde{\varphi}_n^{1,1}(y) = \frac{1}{256}(11n^4 - 2n^3 + 61n^2 + 2n + 4)\hat{\varphi}_n^0(y) + \left[-\frac{1}{24}\check{\varphi}_{n+8}^0(y) - \frac{9}{64}(2n+5)\check{\varphi}_{n+6}^0(y) - \frac{4n^2+11n+3}{12}\check{\varphi}_{n+4}^0(y) + \frac{2n^3+7n^2+31n+30}{64}\check{\varphi}_{n+2}^0(y)\right]_+ + \left[\frac{9}{16}\check{\varphi}_{n+6}^0(y) + \frac{n+1}{2}\check{\varphi}_{n+4}^0(y) - \frac{6n^2+14n+9}{32}\check{\varphi}_{n+2}^0(y)\right]_-, \tag{86}$$

Note that the trend for the eigenenergies in eq. (81) suggests that the terms of the form for $\beta^l\lambda^k$ for $l > 2k$ do not appear. This result is non-trivial, and it is worth investigating further whether this pattern persists through higher orders. If this is the case, it would be interesting to investigate even further to look for reasons behind this.

Perturbative NEAHO Hamiltonian from intertwining relations

NEAHO Hamiltonian can be put in the context of discrete quantum mechanics[10,16]. Let us rename $\hat{H}_A$ from eq. (75) as $\hat{H}_A^{[0]}$. In the context of discrete quantum mechanics, but using our notation, eq. (75) can be written as

$$\frac{\hat{H}_A^{[0]}}{\hbar\omega} = \hat{\mathcal{H}}^{[0]} + \frac{1}{2} = \hat{\mathcal{A}}^{[0]\dagger}\hat{\mathcal{A}}^{[0]} + \frac{1}{2}, \tag{87}$$

where

$$\hat{\mathcal{A}}^{[0]} = i\left(e^{-\frac{i\tilde{\beta}\partial_y}{2}}\sqrt{W^{[0]*}(y)} - e^{\frac{i\tilde{\beta}\partial_y}{2}}\sqrt{W^{[0]}(y)}\right), \tag{88}$$

$$W^{[0]}(y) = \frac{1 + i\tilde{\beta}\sqrt{y^2 + \tilde{\alpha}y^4}}{2\tilde{\beta}^2}, \tag{89}$$

and in this subsection we have chosen not to distinguish operators from their $y-$space representations (this is in the same spirit as writing $\hat{p} = -i\hbar\partial_x$ as a shorthand for $\langle x|\hat{p} = -i\hbar\partial_x\langle x|$). The Hamiltonian is put in the form which allows the use of intertwining relations[10,16], which generate iso-spectral Hamiltonians from appropriate change of operator ordering. The first one from the family is

$$\frac{\widehat{H}_A^{[1]}}{\hbar\omega} = \hat{\mathcal{A}}^{[0]}\hat{\mathcal{A}}^{[0]\dagger} - \frac{1}{2}, \quad (90)$$

and the constant shift $-\hbar\omega/2$ is introduced in order that $\widehat{H}_A^{[1]}$ reduces, in the limit $\beta \to 0$, $\hbar \to 0$, to standard classical AHO Hamiltonian (44). In order to find the next Hamiltonian, one first computes

$$W^{[1]}(y) \equiv \sqrt{W^{[0]}\left(1 - \frac{i\tilde{\beta}}{2}\right) W^{[0]*}\left(1 - \frac{i\tilde{\beta}}{2}\right)} \times \frac{\varphi_0^{[1]}(y - i\tilde{\beta})}{\varphi_0^{[1]}(y)}, \quad (91)$$

where $\varphi_0^{[1]}(y)$ is the ground state of Hamiltonian (90). We expect that $\varphi_0^{[1]}(y)$ do not have a closed form. So let us compute this expression perturbatively in $\lambda_1 \equiv \tilde{\alpha}, \lambda_2 \equiv \tilde{\beta}^2$ keeping orders $\Lambda^{\vec{\mu}} \equiv \lambda_1^{\mu_1}\lambda_2^{\mu_2}$ with $\vec{\mu} = (\mu_1, \mu_2) \leq (2{,}2)$, and such that the unperturbed Hamiltonian is taken to be the standard SHO. Instead of showing the perturbative result for $\varphi_0^{[1]}(y)$, let us directly present

$$W^{[1]}(y) = \frac{1}{4} + \frac{iy}{2\tilde{\beta}} + \frac{1}{2\tilde{\beta}^2} + W_{(1)}^{[1]}(y)\tilde{\alpha} + W_{(2)}^{[1]}(y)\tilde{\alpha}^2 + \mathcal{O}\left(\Lambda^{\vec{\mu} \nleq (2,2)}\right), \quad (92)$$

where

$$W_{(1)}^{[1]}(y) = \left(\frac{3iy}{4} + \frac{iy^3}{4}\right)\frac{1}{\tilde{\beta}} + \frac{3}{8} - \frac{3y^2}{8} + \frac{9iy\tilde{\beta}}{16} + \frac{5\tilde{\beta}^2}{32},$$

$$W_{(2)}^{[1]}(y) = \left(-\frac{51iy}{32} - \frac{11iy^3}{16} - \frac{iy^5}{16}\right)\frac{1}{\tilde{\beta}} - \frac{51}{64} + \frac{5y^4}{32} - \frac{1}{192}i(321y + 198y^3)\tilde{\beta} + \left(-\frac{3}{4} - \frac{5y^2}{64}\right)\tilde{\beta}^2 - \frac{153}{256}iy\tilde{\beta}^3 - \frac{103\tilde{\beta}^4}{512}.$$

In order to obtain $W^{[1]*}(y)$, we set $i \to -i$ in the expression of $W^{[1]}(y)$. Note that we choose the choice $\vec{\mu} \leq (2{,}2)$of truncation simply to give a demonstration of the procedure. The extension to higher order can easily be obtained also by using eq. (42)-(43). With $W^{[1]}(y)$ obtained, one can rewrite as $\widehat{H}_A^{[1]}$

$$\frac{\widehat{H}_A^{[1]}}{\hbar\omega} = \hat{\mathcal{A}}^{[1]\dagger}\hat{\mathcal{A}}^{[1]} + \frac{E_{A1}}{\hbar\omega} - 1, \quad (93)$$

where

$$\hat{\mathcal{A}}^{[1]} = i\left(e^{-\frac{i\tilde{\beta}\partial_y}{2}}\sqrt{W^{[1]*}(y)} - e^{\frac{i\tilde{\beta}\partial_y}{2}}\sqrt{W^{[1]}(y)}\right), \quad (94)$$

and E_{A1} is as given in eq. (81). Note that since $W^{[1]*}$was obtained for $\vec{\mu} \leq (2{,}2)$, the Hamiltonian (93) has to be truncated correspondingly. One can then check that, up to this order, it indeed agrees with its alternative form (90).

One can then obtain another alternative ordering of NEAHO Hamiltonian:

$$\frac{\widehat{H}_A^{[2]}}{\hbar\omega} = \hat{\mathcal{A}}^{[1]}\hat{\mathcal{A}}^{[1]\dagger} + \frac{E_{A1}}{\hbar\omega} - 2. \quad (95)$$

Then one perturbatively computes the ground state wave function $\varphi_0^{[2]}(y)$, and

$$W^{[2]}(y) \equiv \sqrt{W^{[1]}\left(1 - \frac{i\tilde{\beta}}{2}\right) W^{[1]*}\left(1 - \frac{i\tilde{\beta}}{2}\right)} \times \frac{\varphi_0^{[2]}(y - i\tilde{\beta})}{\varphi_0^{[2]}(y)} = \frac{1}{2} + \frac{iy}{2\tilde{\beta}} + \frac{1}{2\tilde{\beta}^2} + W_{(1)}^{[2]}(y)\tilde{\alpha} + W_{(2)}^{[2]}(y)\tilde{\alpha}^2 + \mathcal{O}\left(\Lambda^{\vec{\mu} \nleq (2,2)}\right), \quad (96)$$

where

$$W_{(1)}^{[2]}(y) = \left(\frac{3iy}{2} + \frac{iy^3}{4}\right)\frac{1}{\tilde{\beta}} + \frac{9}{8} - \frac{3y^2}{4} + \frac{15iy\tilde{\beta}}{8}$$

Then is alternatively given as

$$+\frac{7\tilde{\beta}^2}{8},$$

$$W_{(2)}^{[2]}(y) = \left(-\frac{51iy}{8} - \frac{11iy^3}{8} - \frac{iy^5}{16}\right)\frac{1}{\tilde{\beta}} - \frac{255}{64} + \frac{33y^2}{32} + \frac{5y^4}{16} - \frac{1}{192}i(2361y + 630y^3)\tilde{\beta} + \left(-\frac{207}{32} - \frac{37y^2}{32}\right)\tilde{\beta}^2 - \frac{459}{64}iy\tilde{\beta}^3 - \frac{199\tilde{\beta}^4}{64}.$$

Then $\widehat{H}_A^{[2]}$ is alternatively given as

$$\frac{\widehat{H}_A^{[2]}}{\hbar\omega} = \hat{\mathcal{A}}^{[2]\dagger}\hat{\mathcal{A}}^{[2]} + \frac{E_{A2}}{\hbar\omega} - 2 \tag{97}$$

where

$$\hat{\mathcal{A}}^{[2]} = i\left(e^{-\frac{i\tilde{\beta}\partial_y}{2}}\sqrt{W^{[2]*}(y)} - e^{\frac{i\tilde{\beta}\partial_y}{2}}\sqrt{W^{[2]}(y)}\right), \tag{98}$$

Next, one can obtain another alternative ordering:

$$\frac{\widehat{H}_A^{[3]}}{\hbar\omega} = \hat{\mathcal{A}}^{[2]}\hat{\mathcal{A}}^{[2]\dagger} + \frac{E_{A2}}{\hbar\omega} - 3. \tag{99}$$

This should be sufficient to demonstrate the procedure. One may then wish to keep working to obtain also $\widehat{H}_A^{[4]}, \widehat{H}_A^{[5]}, \widehat{H}_A^{[6]}$, etc. Let us now present the energy spectrum that we obtain. For each Hamiltonian, the lowest energy level is labelled by index and subsequent levels are by subsequent numbers. Let us label energy for as $\widehat{H}_A^{[s]}$ as $E_n^{[s]}$. Then

$$E_n^{[0]} = E_{An}, \tag{100}$$

where E_{An} is as given in eq.(81). We obtain the relationship

$$E_n^{[1]} = E_{n+1}^{[0]} - \hbar\omega, \quad E_n^{[2]} = E_{n+1}^{[1]} - \hbar\omega, \quad E_n^{[3]} = E_{n+1}^{[2]} - \hbar\omega, \tag{101}$$

which is so far consistent with the general consideration[10,16]. We expect that this pattern should persist.

Discussion and Conclusion

In this paper, we analyzed NEAHO whose classical potential is of the form eq. (47). We started by making sure that the perturbation method we introduced gives the expected result. This is by comparing the known analysis of NESHO. Then we proceeded by analyzing the NEAHO Hamiltonian (73) by giving a perturbative result of eigenenergies (81) and wavefunctions (82)-(86). We went further by analyzing other orderings of the NEAHO Hamiltonian. These orderings are obtained[10,16] by the application of the iterative procedure. We found, as should be expected, that the eigenenergies for Hamiltonians in the sequence $\widehat{H}^{[0]}, \widehat{H}^{[1]}, \widehat{H}^{[2]}, \cdots$ are related.

It is important to note that although the iterative procedure was already given, obtaining the Hamiltonians in the closed form is not guaranteed. This is because one needs to know the ground state wavefunctions, which do not have a closed form. So perturbation theory provides a way to proceed the analysis. Based on the analysis of the Hamiltonians in our demonstration, the results suggest that the perturbative analysis that we made use is working correctly. So as a future work, one may wish to proceed by using the perturbative analysis to analyze more complicated NEAHO Hamiltonians, for example those whose potentials are of the form

$$V(x) = \frac{1}{2}m\omega^2x^2 + \frac{1}{4}\alpha_1x^4 + \frac{1}{6}\alpha_2x^6 + \frac{1}{8}\alpha_3x^8 + \cdots \tag{102}$$

Eigenenergies and wavefunctions for NEAHO corresponding to this potential can be obtained by using multi-parameter perturbation theory, i.e. by iteratively applying eq. (36).

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