ความเชื่อมโยงในปริภูมิเชิงทอพอโลยีวางนัยทั่วไปอุดมคติ

Connectedness in a set of ideal generalized topological spaces

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บทคัดย่อ

จุดมุ่งหมายของบทความนี้ คือการศึกษาสมบัติของความเชื่อมโยงของเซตในปริภูมิเชิงทอพอโลยีวางนัยทั่วไปอุดมคติ เราได้ให้ลักษณะของเซตเชื่อมโยงในรูปของ μ^* เซตเชื่อมโยงในรูปของ μ^{*s} เซตที่แยกกันได้ในรูปของ μ^* และส่วนประกอบในรูปของ μ^*

คำสำคัญ : ปริภูมิเชิงทอพอโลยีวางนัยทั่วไปอุดมคติ เซตเชื่อมโยงในรูปของ μ^* เซตเชื่อมโยงในรูปของ μ^{*s} เซตที่แยกกันได้ในรูปของ μ^* ส่วนประกอบในรูปของ μ^*

Abstract

The aim of this paper is to study properties of connectedness in ideal generalized topological spaces. Precisely, we provide characterization of μ^*-connected sets, μ^{*s}-connected sets, μ^*-separated sets and μ^*-component.

Keywords : ideal generalized topological spaces, μ^*-connected sets, μ^{*s}-connected sets, μ^*-separated sets, μ^*-component.

Introduction

Kuratowski, (1966) and Vaiyanathaswamy, (1933) introduced the concept of an ideal topological space. They also studied concept of localization theory. An ideal is a nonempty collection of subsets which are closed under heredity and finite union. An ideal I on a topological space (X,τ) is a nonempty collection of subsets of X which satisfies : (1) $A \in I$ and $B \subseteq A$ implies $B \in I$ and (2) $A \in I$ and $B \in I$ implies $A \cup B \in I$.

The notion of connectedness in ideal topological spaces has been introduced by E. Ekici and T. Noiri, (2008). The concept of generalized topological space was introduced by A. Csaszar, (2005). He found the theory of generalized topology to study the extremely elementary character of these classes.

A subfamily μ of the power set P(X) of a nonempty set X is called a generalized topology on X if and only if $\emptyset \in \mu$ and the union of elements of μ belong to μ. We call the pair (X,μ) a generalized topological space on X. The member of μ is called a μ-open set and the complement of a μ-open set is called a μ-closed set.

For $A \subseteq X$, we denote by $c_\mu(A)$ the intersection of all μ-closed sets containing A, and by $I_\mu(A)$ the union of all μ-open sets contained in A. For the ideal I of X, the triple (X,μ,I) is called an ideal generalized topological space.

The purpose of this paper is to introduce and study the union of connectedness in an ideal generalized topological space. We study the notion of properties of

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μ^*-connected sets, μ^{*s}-connected sets, μ^*-separated sets and μ^*-component sets. Next we recall some concepts and definitions which are useful in the results. Some examples are given to illustrate our concepts.

Preliminaries

In this section, we introduce concepts and definitions which are useful in the results.

Definition 1 (S. Modak, 2016) Let (X,μ,I) be an ideal generalized topological space. A mapping $(.)^{*\mu}: P(X) \to P(X)$ is defined as follows :
$A^{*\mu} = \{x \in X : A \cap U \notin I, \forall U \in \mu(x)\}$ for $A \subseteq X$ where $\mu(x) = \{U \in \mu : x \in U\}$.

This mapping is called the local function associated with the ideal I and generalized topology μ.

Definition 2 (S. Modak, 2016) Let (X,μ,I) be an ideal generalized topological space. The set operator $c^{*\mu}$ is called a generalized $*$-closure and is defined as $c^{*\mu}(A) = A \cup A^{*\mu}$ for $A \subseteq X$. We denote by $\mu^*(\mu,I)$ the generalized structure, generated by $c^{*\mu}$, that is $\mu^*(\mu,I) = \{U \subseteq X : c^{*\mu}(X \setminus U) = X \setminus U\}$. $\mu^*(\mu,I)$ is called a $*$-generalized structure which is finer than μ. The elements of $\mu^*(\mu,I)$ are called μ^*-open and the complement of μ^*-open set are called μ^*-closed. We simply write $A^{*\mu}$ for $A^{*\mu}(\mu,I)$.

Theorem 3 (S. Modak, 2016) The set operator $c^{*\mu}$ satisfies the following conditions :

(1) $A \subseteq c^{*\mu}(A)$ for $A \subseteq X$,

(2) $c^{*\mu}(\varnothing) = \varnothing$ and $c^{*\mu}(X) = X$,

(3) $c^{*\mu}(A) \subseteq c^{*\mu}(B)$, if $A \subseteq B \subseteq X$,

(4) $c^{*\mu}(A) \cup c^{*\mu}(B) \subseteq c^{*\mu}(A \cup B)$,

(5) $c^{*\mu} \in \Gamma(X)$ where $\Gamma(X)$ is the collection of all mappings having the property monotony (i.e. such that $A \subseteq B$ implies $\gamma(A) \subseteq \gamma(B)$).

Throughout the paper μ will represent a generalized topological spaces such that $\varnothing, X \in \mu$ and the union of elements of μ belong to μ.

Theorem 4 Let (X,μ) be a generalized topological space. Then $\mu^*(\mu,I)$ is a generalized topological space.

Proof. Since $\mu^*(\mu,I) = \{U \subseteq X : c^{*\mu}(X \setminus U) = X \setminus U\}$, we have $\varnothing, X$ are in $\mu^*(\mu,I)$. Next, we will show that if $A_n \in \mu^*$ for all $n \in \Delta$, then $\bigcup_{n\in\Delta} A_n \in \mu^*$. Let $A_n \in \mu$ for all $n \in \Delta$. Since $X \setminus \bigcup_{n\in\Delta} A_n \subseteq c^{*\mu}\left(X \setminus \bigcup_{n\in\Delta} A_n\right)$, we have to show that $c^{*\mu}\left(X \setminus \bigcup_{n\in\Delta} A_n\right) \subseteq X \setminus \bigcup_{n\in\Delta} A_n$. Consider

$$
\begin{aligned}
c^{*\mu}\left(X \setminus \bigcup_{n\in\Delta} A_n\right) &= c^{*\mu}\left(\bigcap_{n\in\Delta}(X \setminus A_n)\right) \\
&= \left(\bigcap_{n\in\Delta}(X \setminus A_n)\right) \cup \left(\bigcap_{n\in\Delta}(X \setminus A_n)\right)^{*\mu} \\
&\subseteq \left(\bigcap_{n\in\Delta}(X \setminus A_n)\right) \cup \left(\bigcap_{n\in\Delta}(X \setminus A_n)^{*\mu}\right) \\
&\subseteq \bigcap_{n\in\Delta}\left[(X \setminus A_n) \cup (X \setminus A_n)^{*\mu}\right] \\
&= \bigcap_{n\in\Delta}\left[c^{*\mu}(X \setminus A_n)\right] \\
&= \bigcap_{n\in\Delta}(X \setminus A_n) \\
&= \bigcap_{n\in\Delta}(X \cap A_n^c) \\
&= X \cap \left(\bigcap_{n\in\Delta} A_n^c\right) \\
&= X \cap \left(\bigcup_{n\in\Delta} A_n\right)^c \\
&= X \setminus \bigcup_{n\in\Delta} A_n.
\end{aligned}
$$

A subset A of an ideal generalized topological space (X,μ,I) is said to be μ^*- dense if $c^{*\mu}(A) = X$. An ideal generalized topological space (X,μ,I) is said to be μ^*- hyperconnected if A is μ^*- dense for every nonempty μ-open subset A of X. A generalized topological space (X,μ) is said to be μ-hyperconnected if every pair of nonempty μ-open sets of X has nonempty intersection.

So that if (X,μ,I) is an ideal generalized topological space and K is a subset of X, then (K,μ_K,I_K), where μ_K is the relative generalized topological on K and $I_K = \{K \cap J : J \in I\}$ is an ideal on K.

Lemma 5 Let (X,μ,I) be an ideal generalized topological space and $M\subseteq K\subseteq X$.

Then $M^{*\mu}(\mu_K,I_K)=M^{*\mu}(\mu,I)\cap K$.

Proof. Let $x\in M^{*\mu}(\mu_K,I_K)$. Then $x\in K$. Suppose that $x\notin M^{*\mu}(\mu,I)$. Then there exists a μ-open set U such that $U\cap M\in I$. Since $U\cap K\in\mu_K(x)$. Then $(U\cap K)\cap M=(U\cap M)\cap K\in I_K$. So $x\in M^{*\mu}(\mu,I)$ and $x\in K$. This is a contradiction. Thus $x\in M^{*\mu}(\mu,I)\cap K$. Conversely, let $y\in M^{*\mu}(\mu,I)\cap K$. Assume that $y\notin M^{*\mu}(\mu_K,I_K)$. Then there exists $G\in\mu_K(y)$ such that $G\cap M\in I_K$. Since $G\in\mu_K(y)$, there exists μ-open set U such that $G=U\cap K$. Since $M\subseteq K$ and $I_K\subseteq I$, $(U\cap K)\cap M\in I$ So $y\in U\cap M\in I$. This is a contradiction. Hence $y\in M^{*\mu}(\mu_K,I_K)$.

Definition 6 Let (X,μ,I) be an ideal generalized topological space and $M\subseteq K\subseteq X$. A generalized $*$-closure on K is defined as $\mathbf{c}_K^{*\mu}(M)=\mathbf{c}^{*\mu}(M)\cap K$.

Connectedness in a set of ideal generalized topological spaces

In this section, we introduce the concept of μ^*-connected sets, μ^{*s}-connected sets, μ^*-separated sets and μ^*-component. Moreover, we study characterization of μ^*-dense and μ^*-hyperconnected.

Definition 1 An ideal generalized topological space (X,μ,I) is called μ^*-connected if X cannot be written as disjoint union of a nonempty μ-open set and a nonempty μ^*-open set.

Definition 2 A generalized topological space (X,μ) is called μ-connected if X cannot be written as disjoint union of two nonempty μ-open sets.

Example Let $X=\{a,b,c,d\}$,
$\mu=\{\emptyset,X,\{a,b\},\{b,c\},\{a,b,c\}\}$, $I=\{\emptyset,\{b\}\}$.
Then $\mu^*=\{\emptyset,X,\{a\},\{c\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\},\{a,c,d\},\{b,c,d\}\}$.
Therefore, (X,μ,I) is μ^*-connected.

Definition 3 A subset A of an ideal generalized topological space (X,μ,I) is called μ^*-connected if (A,μ_A,I_A) is μ^*-connected.

Example Let $X=\{a,b,c,d\}$,
$\mu=\{\emptyset,X,\{a,b\},\{b,c\},\{a,b,c\}\}$, $I=\{\emptyset,\{b\}\}$ and
$A=\{a,b,d\}$. Then $\mu_A=\{\emptyset,A,\{b\},\{a,b\}\}$, $I_A=\{\emptyset,\{b\}\}$ and $\mu_A^*=\{\emptyset,A,\{a\},\{a,b\},\{a,c\}\}$. Therefore, (A,μ_A,I_A) is μ^*-connected.

Remark 4 (1) Generally, it is known that every μ-hyperconnected generalized topological space is μ-connected, but not conversely.

(2) Every μ^*-hyperconnected generalized topological space is μ^*-connected, but not conversely.

(3) Every μ^*-hyperconnected generalized topological space is μ-hyperconnected, but not conversely.

(4) Every μ^*-connected generalized topological space is μ-connected, but not conversely.

Example (1) Let $X=\{a,b,c\}$ and $\mu=\{\emptyset,X,\{a\},\{b\},\{a,b\}\}$. Then the space (X,μ) is μ-connected but (X,μ) is not μ-hyperconnected.

(2) Let $X=\{a,b,c,d\}$,
$\mu=\{\emptyset,X,\{a\},\{c\},\{a,b\},\{a,c\},\{a,b,c\},\{a,c,d\}\}$ and $I=\{\emptyset,\{b\}\}$. Then the space (X,μ,I) is μ^*-connected but (X,μ,I) is not μ^*-hyperconnected.

(3) Let $X=\{a,b,c,d\}$,
$\mu=\{\emptyset,X,\{a,b\},\{a,c\},\{a,b,c\}\}$ and $I=\{\emptyset,\{a\},\{d\},\{a,d\}\}$. Then the space (X,μ) is μ-hyperconnected but (X,μ,I) is not μ^*-hyperconnected.

(4) Let $X=\{a,b,c,d\}$,
$\mu=\{\emptyset,X,\{c\},\{a,b\},\{a,c\},\{a,b,c\}\}$ and $I=\{\emptyset,\{b\}\}$. Then the space (X,μ) is μ-connected but (X,μ,I) is not μ^*-connected.

Definition 5 Nonempty subsets M,K of an ideal generalized topological space (X,μ,I) are called μ^*-separated if $\mathbf{c}^{*\mu}(M)\cap K=M\cap\mathbf{c}_\mu(K)=\emptyset$.

Definition 6 Let (X,μ,I) be an ideal generalized topological space and $(\mathbf{Y},\mu_Y,I_Y)$ be a subspace of X. Nonempty subsets M,K of $(\mathbf{Y},\mu_Y,I_Y)$ are called μ^*-separated if $\mathbf{c}_Y^{*\mu}(M)\cap K=M\cap\mathbf{c}_Y^{\mu}(K)=\emptyset$.

Theorem 7 If M and K are μ^*-separated sets of X and $M\cup K\in\mu$, then M is μ-open and

K is μ^* - open.

Proof. Let M and K be μ^*- separated sets of X. Then we get that $K \subseteq X \setminus c^{*\mu}(M)$. Since $c^{*\mu}(M)$ is μ^* - closed and $K = (M \cup K) \cap (X \setminus c^{*\mu}(M))$. K is a μ^* - open set. We get that M is a μ - open set likewise.

Theorem 8 Let (X,μ,I) be an ideal generalized topological space and $M,K \subseteq \mathbf{Y} \subseteq X$. M,K are μ^*- separated in $\mathbf{Y}$ if and only if M,K are μ^*- separated in X.

Proof. Suppose that M,K are μ^*- separated in $\mathbf{Y}$. By Lemma 2.5, we get that

$c_Y^{*\mu}(M) \cap K = (c^{*\mu}(M) \cap \mathbf{Y}) \cap K = c^{*\mu}(M) \cap K.$

Thus

$c^{*\mu}(M) \cap K = c_Y^{*\mu}(M) \cap K = \varnothing = (M \cap c_Y^{\mu}(K)) \cap K$

$= M \cap (c_Y^{\mu}(K) \cap \mathbf{Y}) = M \cap ((\mathbf{Y} \cap c_{\mu}(K)) \cap \mathbf{Y})$

$= M \cap c_{\mu}(K)$. Therefore M,K are μ^*- separated in X. Conversely, suppose that subsets M,K are μ^*- separated in X. Since $M,K \subseteq \mathbf{Y}$ and by Definition 3.5, we get that $c^{*\mu}(M) \cap K = M \cap c_{\mu}(K) = \varnothing$. So

$c_Y^{*\mu}(M) \cap K = c^{*\mu}(M) \cap K = \varnothing = M \cap c_{\mu}(K)$

$= (M \cap \mathbf{Y}) \cap c_{\mu}(K) = M \cap (c_{\mu}(K) \cap \mathbf{Y})$

$= M \cap c_Y^{*\mu}(K)$. Thus $c_Y^{*\mu}(M) \cap K = \varnothing = M \cap c_Y^{\mu}(\kappa)$.

Therefore M,K are μ^*- separated in $\mathbf{Y}$.

Definition 9 A subset M of an ideal generalized topological space (X,μ,I) is called μ^{*s}- connected if M is not the union of two μ^*- separated sets in (X,μ,I).

Theorem 10 Let (X,μ,I) be an ideal generalized topological space. If M is a μ^{*s}- connected set in X and H,K are μ^*- separated sets in X with $M \subseteq H \cup K$, then either $M \subseteq H$ or $M \subseteq K$.

Proof. Let M be a μ^{*s}- connected set in X and H,K be μ^*- separated sets in X with $M \subseteq H \cup K$. Then $M = M \cap (H \cup K) = (M \cap H) \cup (M \cap K)$ and $c^{*\mu}(H) \cap K = H \cap c_{\mu}(K) = \varnothing$. Thus $(M \cap K) \cap c^{*\mu}(M \cap H) \subseteq K \cap c^{*\mu}(H) = \varnothing$. Likewise, we have $(M \cap H) \cap c_{\mu}(M \cap K) = \varnothing$. If $M \cap H \neq \varnothing$ and $M \cap K \neq \varnothing$, then $M \cap H$ and $M \cap K$ are μ^* - separated sets in X, so M is not a μ^*- separated set. This is a contradiction. Thus either $M \cap H$ or $M \cap K$ are empty. Assume that $M \cap H = \varnothing$. Then $M = M \cap K$ implies that $M \subseteq K$. In a similar way, we have that $M \subseteq H$.

Theorem 11 Let (X,μ,I) be an ideal generalized topological space. If M is a μ^{*s}- connected set and $M \subseteq K \subseteq c^{*\mu}(M)$, then K is μ^{*s}- connected.

Proof. Assume that K is not μ^{*s}- connected.

Then there exist μ^*- separated sets A and B such that $K = A \cup B$ i.e. A,B are nonempty and $c^{*\mu}(A) \cap B = A \cap c_{\mu}(B) = \varnothing$. By Theorem 3.10, we get that either $M \subseteq A$ or $M \subseteq B$.

Assume that $M \subseteq A$. Then $c_{\mu}(M) \subseteq c_{\mu}(A)$ and $B = B \cap c^{*\mu}(M) \subseteq B \cap c_{\mu}(M) = \varnothing$. Thus B is an empty set. This is a contradiction. Suppose $M \subseteq B$.

Then we have $c^{*\mu}(M) \subseteq c^{*\mu}(B)$ and $A = A \cap c^{*\mu}(M) \subseteq A \cap c^{*\mu}(B) = \varnothing$. Thus A is nonempty. This is a contradiction. Therefore, K is μ^{*s}- connected.

Corollary 12 Let (X,μ,I) be an ideal generalized topological space. If M is a μ^{*s}- connected set, then $c^{*\mu}(M)$ is μ^{*s}- connected.

Theorem 13 Let (X,μ,I) be an ideal generalized topological space. If $\{M_n : n \in \Omega\}$ is nonempty family of μ^{*s}- connected sets with $\bigcap_{n\in\Omega} M_n \neq \varnothing$, then $\bigcup_{n\in\Omega} M_n$ is a μ^{*s}- connected set.

Proof. Suppose that $\bigcup_{n\in\Omega} M_n$ is not a μ^{*s}- connected set. Then we have that $\bigcup_{n\in\Omega} M_n = A \cup B$ where A and B are μ^*- separated sets. Since $\bigcap_{n\in\Omega} M_n \neq \varnothing$, we have a point $x \in \bigcap_{n\in\Omega} M_n$. Thus $x \in M_n$ and $M_n \subseteq A \cup B$ for all $n \in \Omega$. It follows from $A \cap B = \varnothing$ that either $x \in A$ or $x \in B$. Case $x \in A$. For any $n \in \Omega, M_n \cap A \neq \varnothing$. Using Theorem 3.10, we have $M_n \subseteq A$ or $M_n \subseteq B$. So $M_n \subseteq A$ for all $n \in \Omega$ and then $\bigcup_{n\in\Omega} M_n \subseteq A$. This implies that $B \neq \varnothing$. This is a contradiction. In the same way for a case $x \in B$, we have $A = \varnothing$. This is a contradiction. Hence $\bigcup_{n\in\Omega} M_n$ is a μ^{*s} - connected set.

Definition 14 Let (X,μ,I) be an ideal generalized topological space and $x \in X$. The union of all μ^{*s} - connected subsets of X containing x is called the μ^{*} - component of X containing x

Theorem 15 Each μ^{*} - component of an ideal generalized topological space (X,μ,I) is a maximal μ^{*s} - connected set.

Proof. Let $x \in X$. Suppose that c_x is μ^{*} - component of X such that $x \in c_x$ so $c_x = \bigcup_{j\in J}\{M_j \subseteq X\}$ where M_j is a μ^{*s} - connected set containing x. Then $\bigcap_{j\in J}\{M_j \subseteq X : x \in M_j\} \neq \varnothing$. Since M_j is μ^{*s} - connected for all j□J, theorem 3.13, implies that c_x is μ^{*} - connected. Next, let $A \subseteq X$ and A be μ^{*s} - connected such that $c_x \subseteq A$. Then $x \in A$, by definition of c_x we have $A \subseteq c_x$. Thus $c_x = A$. Therefore, c_x is a maximal μ^{*s} - connected set of X.

Theorem 16 The set of all distinct μ^{*} - component of an ideal generalized topological space (X,μ,I) forms a partition of X.

Proof. Let M and K be two distinct μ^{*} - components of X. Suppose that $M \cap K \neq \varnothing$. By Theorem 3.13, $M \cup K$ is μ^{*s} - connected in X. Since $M \subseteq M \cup K$. Then M is not maximal. Thus M and K are disjoint. Since M and K are distinct μ^{*} - components of X. By Theorem 3.15, we get that M and K are maximal μ^{*s} - connected set of X. So, $M = X \setminus K$. Hence $M \cup K = (X \setminus K) \cup K = X$.

Theorem 17 Each μ^{*} - component of an ideal generalized topological space (X,μ,I) is μ^{*} - closed.

Proof. Let A be μ^{*} - component of X. Then, by Theorem 3.15, and Corollary 3.12, we get that A is maximal μ^{*s} - connected and $c^{*\mu}(A)$ is μ^{*s} - connected. Thus $A = c^{*\mu}(A)$. This implies that A be μ^{*} - closed.

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